Conformal Gravity

Kit Colwell

December 2, 2011

Conformal Transformations

General relativity (GR), as formulated by Einstein in the early 20^{th} century, is spoken entirely in the language of differential geometry, the study of curves in space (and space itself) formalized by a host of mathematicians in the 1800's, most notably B. Riemann and C. F. Gauss. The wonderful utility of this theory is its appeal to the invariance of physics under any arbitrary change of local coordinates, known as a general coordinate transformation (GCT). Much of the work done in GR since its inception has been the practice of inventing possible metrics for spacetime, and examining physical behavior from the perspective of differing coordinate systems, though of course the physics itself should not depend on any local coordinate chart choice. Names have been made for physicists and mathematicians who discover a particularly convenient coordinate system for a given metric.

Of great use is a class of GCTs known as conformal transformations (CT), which are defined as preserving the oriented angles between curves at each point. The original and still primary use of such maps is in the complex plane $\mathbb{C}$, where one of the most important theorems of complex analysis, the Riemann Mapping Theorem, states that any non-empty open simply connceted proper subset of $\mathbb{C}$ admits a bijective conformal map to the open unit disk in $\mathbb{C}$. This is of particular use in physics when coupled to another result from complex analysis: any harmonic function (satisfying Laplace's equation $\nabla^2 f = 0$) on an open set will remain harmonic under a conformal transformation. Hence, harmonic functions can be defined equivalently on any open set of $\mathbb{C}$. We also have the results that for an open set $U \subset \mathbb{C}$, a function $f : U \to \mathbb{C}$ is conformal if and only if it is holomorphic and $f'(z) \neq 0$ on U, and any conformal map from the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (conformally equivalent to a sphere through stereographic projection) to itself must be a Möbius transformation

$$f(z) = \frac{az + b}{cz + d}.$$

The group of such automorphisms of $\hat{\mathbb{C}}$, $\mathrm{Aut}(\hat{\mathbb{C}})$, is called the Möbius group, and is isomorphic to the projective group $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm I\}$. From this, we can see that the Möbius group is a 3-dimensional complex semisimple non-compact Lie group.[4] Incidentally, the Möbius group shows up in numerous other places in physics: it is also isomorphic to the proper, orthochronous (restricted) Lorentz group $SO^+(1,3)$, a relationship that we will examine later on, which also connects deeply with spin groups and the use of Weyl sinors in the Dirac equation and supersymmetric theories. It also forms the group of orientation-preserving isometries of hyperbolic 3-space H^3 (and if restricted over the reals to PSL(2,$\mathbb{R}$), hyperbolic 2-space H^2): if we use the Poincaré ball model and identify the unit ball in $\mathbb{R}^3$ with H^3, then we can think of the Riemann sphere as the "conformal boundary" of H^3. This was one of the first observations leading to the AdS/CFT correspondance conjecture, which will also be addressed later. It is fairly straightforward to generalize the Möbius group to higher dimensions, so that $\text{Möb}(n) \cong \text{SO}^+(1, n+1)$ is the group of all orientation-perserving conformal isometries of the sphere S^n.[2]

Of particular use to us will be the classification of symmetries related to conformal transformations, since these symmetries may give us Killing vectors in the extension to differential geometry. Liouville has a theorem about conformal mappings in Euclidean space stating that any smooth conformal mapping on a subset of $\mathbb{R}^n$ with $n > 2$ can be expressed as a composition of translations, similarities, orthogonal transformations, and inversions (notice that this does not hold for $n = 2$: the Riemann Mapping Theorem asserts that all simply connected planar domains are conformally equivalent). This theorem extends naturally to any space of dimension $n > 2$, so conformal transformations are generated by translations (parabolic transforms), rotations (elliptic transforms), dilations or scalings (hyperbolic or "homothetic" transforms), and so-called "special conformal transformations" (loxodromic transforms), which amount to reflections and inversions through a sphere. The group and algebra structure of these transformations are discussed later on.

Needless to say, the universality of conformal transformations, similarities, and symmetries suggests that they are likely of great importantance as a fundamental concept in physical theories that may go beyond our current understanding of our universe, so special attention should be given to their study, classification, and use.

Conformal Geometry

This structure is easily extended to the topic of differential geometry, with a slight modification of terminology. A *conformal manifold* is a differentiable manifold equipped with an equivalence class of Riemannian or pseudo-Riemannian metrics, where

$$h \sim g \quad \text{iff} \quad h_{\mu\nu}(x) = \lambda^2(x) g_{\mu\nu}(x)$$

In other words, the two conformal metrics are identical up to a Weyl transformation, a local change of scale. While it is possible to imagine transformations which are conformal (angle preserving) but not Weyl (scale-changing), they are bizzare and uncommon, so the terms "Weyl" and "conformal" transformations are often used interchangeably. A conformal metric is conformally flat if one of its representative metrics is flat (Riemann curvature tensor vanishes). Sometimes we might only be able to find a metric in the conformal class that is flat in an open neighborhood of each point, or locally. In two dimensions, every conformal metric is locally conformally flat. In higher dimensions, we may concern ourselves only with the trace-free part of the curvature tensor, since this part is not concerned with how volumes change, but rather only how a shape is distorted under a GCT. In 3 dimensions, a necessary and sufficient condition for conformal flatness is the vanishing of the Cotton tensor:

$$C_{\mu\nu\rho} = \nabla_\alpha W^\alpha{}_{\mu\nu\rho} = -\frac{1}{2}\left[\nabla_\rho R_{\mu\nu} - \nabla_\nu R_{\mu\rho} + \frac{1}{2(n-1)}\left(g_{\mu\rho}\nabla_\nu R - g_{\mu\nu}\nabla_\rho R\right)\right]$$

while if $n \geq 4$, it is the Weyl tensor that must vanish:

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{n-2}\left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{2}{(n-1)(n-2)}Rg_{\mu[\rho}g_{\sigma]\nu}$$

Under a conformal transformation, the Weyl tensor is completely invariant (the Cotton tensor changes by a total derivative). The only other conformally invariant tensor which is algebraically independent of the Weyl tensor is the Bach tensor:

$$B_{\mu\nu} = 2\nabla^\alpha\nabla^\beta W_{\alpha\mu\nu\beta} + W_{\alpha\mu\nu\beta}R^{\alpha\beta},$$

which arises often in the equations of motion for conformal gravity.

In 1913, swedish physicist G. Nordström built a theory of gravity incorporating an identically vanishing Weyl tensor which showed great theoretical promise, even to Einstein, who at this time was developing his general theory of relativity. Alas, experimental evidence supports instead the GR vanishing of the *Ricci tensor* and scalar in vacuum, and not the *Weyl* tensor and Ricci *scalar* as suggested by Nordström. Nevertheless, having a metric that is scale-invariant is an appealing idea, in that it seems natural that no particular length scale should be preferred by the universe (except perhaps the Planck scale, but of course that introduces quantum effects that might not best be dealt with in the language of manifold diffeomorphisms, and conformal symmetry may be badly broken anyway).

Conformal symmetry can also be of great use in deciding on coordinates to use even in non-conformally invariant spacetimes. For example, the use of light-cone diagrams in special relativity's Minkowski space is intuitively easy to grasp and understand. It would be wonderful if we could use these same diagrams in the setting of general relativity, where our metric may be more complicated, and light rays may no longer be at 45° angles. To resolve this issue, we may examine a locally conformally equivalent metric in a *conformal* or *Penrose diagram*, where the actual metric is transformed conformally such that null paths once again form 45°, allowing the causal structure to be more transparent. The invariance of null paths under a conformal transformation plays a great role in examining massless particle theories, for after all, they must follow these paths. By transforming a given metric to one in which timelike and spatial infinities are only a finite coordinate away allows us to draw a diagram of the entire spacetime in a compact setting, illuminating the entire causal history.

Incidentally, when we apply a local gauge transformation to massless particles, it has exactly the same structure as introducing a spin connection that maintains local conformal invariance. This emerges because massless particles move on the light cone, and these are left invaraint not only by Poincaré symmetries but the full 15-dimensional conformal group $SO(4,2)$, whose universal covering group is $SU(2,2)$. Since we know that $SU(2,2)$

is generated by the 15 Dirac matrices $\{\gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu}\}$, its fundamental representation is a fermionic field, and a 4-component Dirac spinor is irreducible under the conformal group just as it is under the Poincaré group.[7]

Conformal Symmetries

Let's more closely examine the symmetries generated by the conformal group in 4 dimensions (it can easily be generalized to more). We first pick out a representation for the generators of the algebra, starting with the standard ones for the Poincaré group and adding the additional conformal transformation generators:

$$\begin{aligned} M_{\mu\nu} &= i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) \\ P_\mu &= -i\partial_\mu \\ D &= -ix_\mu \partial^\mu \\ K_\mu &= i\left(x^2\partial_\mu - 2x_\mu x_\nu \partial^\nu\right) \end{aligned}$$

where $M_{\mu\nu}$ are the standard Lorentz group generators (rotations and boosts), P_μ generates translations, D generates dilations (sometimes called "dilatations"), and K_μ generates the special conformal transformations that take

$$x^\mu \to \frac{x^\mu - a^\mu x^2}{1 - 2a \cdot x + a^2 x^2}$$

The algebra structure is given by the commutators:

$$\begin{aligned} [P_\rho, M_{\mu\nu}] &= i\left(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\right) \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}\right) \\ [D, K_\mu] &= -iK_\mu \\ [D, P_\mu] &= iP_\mu \\ [K_\mu, P_\nu] &= 2i\eta_{\mu\nu}D - 2iM_{\mu\nu} \\ [K_\mu, M_{nu\rho}] &= i\left(\eta_{\mu\nu}K_\rho - \eta_{\mu\rho}K_\nu\right) \end{aligned}$$

The first two commutators along with $[P_\mu, P_\nu] = 0$ define the Poincaré algebra $\mathfrak{iso}^+(1,3)$ ($\mathfrak{i}$ meaning "inhomogeneous"), which is the 10 dimensional algebra of isometries of Minkowski space. The Poincaré group is the affine group of the Lorentz group (i.e., the semidirect product of translations with the Lorentz group: $\mathrm{ISO}^+(1,3) \cong \mathbb{R}^{1,3} \rtimes \mathrm{SO}^+(1,3)$). The geometry of Minkowski space is defined by the Poincaré group: it is a homogeneous space for the group. We might also introduce two other maximally symmetric spacetimes at this point: de Sitter space dS_n and anti-de Sitter space AdS_n, whose topologies and isomorphisms as quotient groups are $\mathbb{R}^1 \times S^{n-1}$, $O(1,n)/O(1,n-1)$ and $S^1 \times \mathbb{R}^{n-1}$, $O(2,n-1)/O(1,n-1)$ respectively (at least with $\mathcal{PT}$ symmetry; if we don't have these, the spaces become quotients of spin groups). We can see that they bear a great resemblance to a sphere, whose quotient structure for comparison is $S^n = O(n-1)/O(n)$. We will return to these spacetimes again later on.

The 10 generators of the Lorentz group (algebra elements P_μ and $M_{\mu\nu}$) are in one-to-one correspondance with the Killing vectors that can be obtained from Killing's equation

$$\nabla_\mu \chi_\nu + \nabla_\nu \chi_\mu = 0$$

using the Minkowski metric $\eta_{\mu\nu}$. If we wish to extend this to the conformal group, we will need 5 additional Killing vectors for dilations and special conformal transformations. These are none other than the generators D and K_μ. Incidentally, another way to determine these additional Killing vectors is to use the fact that if

$$\overline{g}_{\mu\nu} = e^{\alpha(x)} g_{\mu\nu},$$

and χ^μ is a Killing vector of the original metric, then it is a conformal Killing vector for the barred metric; that is, it satisfies the conformal Killing equation

$$\nabla_\mu \chi_\nu + \nabla_\nu \chi_\mu = \left(\nabla_\sigma \alpha\right) \chi^\sigma g_{\mu\nu}$$

Solving this, we find our most general conformal Killing vector in n dimensions (any signature):

$$\chi^\mu = a^\mu + \omega^{\mu\nu} x_\nu + b x^\mu + c_\nu \left(2x^\mu x^\nu - \eta^{\mu\nu} x^2\right)$$

with $(a^\mu, \omega^{\mu\nu}, b, c_\nu)$ a total of $(n+1)(n+2)/2$ parameters: n translations, $n(n-1)/2$ Lorentz transformations, 1 dilation, and n special conformal transformations. If the dimension of the set of conformal Killing vectors is 15, then the space is conformally flat. If not, then the maximal dimension is 7.[10] From Noether's theorem, these continuous symmetries should have some conserved quantity associated with them, and in fact we know that 4-momentum and angular momentum are conserved from the Poincaré subalgebra symmetry. What about the explicitly conformal parts? Well, a problem arises when we consider the commutator $[D, P_\mu] = iP_\mu$: this tells us that

$$e^{-i\alpha D} P^2 e^{i\alpha D} = e^{-2\alpha} P^2$$

so that if we have some 1-particle state of mass m, $P^2|P\rangle = m^2|P\rangle$ and $P^2|\overline{P}\rangle = e^{2\alpha} m^2 |\overline{P}\rangle$ (we are using the fact that P^2 is a quadratic Casimir invariant of the Poincaré group to label mass states). But if local scale invariance is not broken, we would have $e^{-i\alpha D}|0\rangle = |0\rangle$, indicating that $|P\rangle$ and $|\overline{P}\rangle$ belong to the same Hilbert space, implying that the mass spectrum is either continuous, or all masses vanish! This is not what we see in nature, so we must conclude that P^2 is not a Casimir invariant for the full conformal group, and that conformal symmetry must be broken, at least slightly. Hence, in our own universe, at least at low energy, we won't have conserved Noether currents corresponding to the dilation (or special conformal transformation not shown here) symmetry. Even without an explicit conformal symmetry, we can still put it to good use in a theory of gravity, show that a conformal symmetry can help explain rotation curve anomalies for galaxies, and perhaps hint at a possible theory of quantum gravity.

Gravity

The simplest metric theory known to have conformal symmetry is a "Weyl-squared" theory, which has as its action

$$I = \int d^4x \sqrt{-g} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}.$$

Putting this action in terms of the curvatures we usually see in the Einstein theory, this becomes

$$I = \int d^4x \sqrt{-g} \left[R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \right]$$

which unfortunately contains fourth order products and derivatives of the metric. Why this should be problematic will be discussed later. This result can be put in a more useful form following the idea of C. Lanczos to use an infinitesimal conformal transformation. The result, after sweeping away some total divergences in the integral, is

$$\int d^4x \sqrt{-g} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} = \int d^4x \sqrt{-g} \left[R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right]$$

a much cleaner result, allowing for equations of motion consistent with $R_{\mu\nu} = 0$. If we vary the metric, we obtain

$$\delta \int d^4x \sqrt{-g} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} = \int d^4x \sqrt{-g} W_{\mu\nu} \delta g^{\mu\nu}$$

where $W_{\mu\nu}$ is a complicated expression involving the Ricci tensor, scalar, and their derivatives. In a space with matter, we have $W_{\mu\nu} \propto T_{\mu\nu}$.[12] By setting the variation to zero, we have an equation of motion known as the Bach equation (the Bach tensor $B_{\nu\rho}$ was introduced earlier):

$$B_{\nu\rho} = 2\nabla_\mu \nabla_\sigma W^\mu{}_{\nu\rho}{}^\sigma + W^\mu{}_{\nu\rho}{}^\sigma R_{\mu\sigma} = 0$$

which has conformally flat metrics as its solutions. If we assume a static, spherically symmetric metric in vacuum

$$ds^2 = -b(\rho)dt^2 + a(\rho)d\rho^2 + \rho^2 d\Omega^2$$

it can be shown [8] that this is conformally equivalent to the line element

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2$$

and that using the actual form of the Weyl tensor, we may solve

$$B(r) = 1 - \frac{\beta(2 - 3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2$$

without any approximation at all. The relevant components of the Weyl tensor are proportional to $\beta(2 - 3\beta\gamma + \gamma r)/r$, which is conformally flat when $\beta = 0$. The solution then appears to represent a massive body embedded in a conformally flat spherically symmetric space. This means that conformal flatness is broken by the mere presence of the massive body, even at infinity, suggesting a gravitational origin for intertial mass, and showing that the theory allows for the breaking of conformal symmetry in its own solution.[8]

Note that if $\gamma = k = 0$ and $\beta = m$, this is exactly the Schwartzchild metric from the ordinary flat space Einstein-Hilbert theory. If we don't set $k = 0$, then this is the Schwartzchild solution in a de Sitter background with $R = -12k$, where in the Einstein theory such a term could only come from a cosmological constant. In the Weyl theory, the de Sitter solution is a vacuum solution, and so does not involve any cosmological constant at all. As such, we should think of the parameter γ as measuring the difference between a Weyl theory and an Einstein theory with a cosmological constant, so that for small γ, the two should be very much alike. For small r, both have a $1/r$ gravitational potential consistant with Newtonian gravity. At cosmological scales, the cosmological constant term dominates. However, at intermediate scales, we expect to see variations between the theories due to the γr term in the Weyl metric.[7]

One of these variations could be the observed discrepancy between velocities of galactic rotation and those predicted using GR. It has been noticed that the speeds of stars towards the edges of galaxies are much greater than what they should be considering the amount of observed mass within the galaxy. This was one of the primary motivations for the concept of dark matter, a type of non-luminous gravitating matter that seems to permeate the area around galaxies in enormous spherical halos. However, if we expect a linear potential to be present at large scales due to conformal effects from using the Weyl action, we should see a potential around a star of

$$V^*(r) = -\frac{\beta^* c^2}{r} + \frac{\gamma^* c^2 r}{2}$$

per unit solar mass, with β^* the familiar $M_\odot G/c^2 = 1.48 \times 10^5$ cm. Mannheim and O'Brien [9] fit this to the observed rotation patterns of 11 (original paper, now 134 at the time of this writing) galaxies and found a good fit with $\gamma^* = 5.42 \times 10^{-41}$ cm^{-1} with no recourse to dark matter whatsoever.

Now, there is a caveat to this seemingly amazing result. The Weyl gravity theory, being quadratic in the curvature tensor, is actually fourth order in the metric derivatives (linearize the metric and you'll see $\Box^2 h_{\mu\nu}$ showing up in the equations of motion). Such higher (than 2) derivative theories have a built in instability first studied by Ostrogradski: the Hamiltonian obtained in the canonical way will be *linear* in $n-1$ conjugate momenta if the Lagrangian depends on the nth derivative of a coordinate.[13] Hence, we expect to see Hamiltonians that are generally bounded below if the Lagrangian depends on no more than the first derivative of the metric. In the Weyl theory, this dependence on the fourth derivative implies the existence of ghosts with unbounded negative energy states that would be continuously popping out of the vacuum. One attempt to remedy the situation is to employ the Hawking-Hertog formalism and integrate out the ghosts in a Euclidean path integral technique[3]. Another uses the Pais-Uhlenbeck oscillator model to determine a Hilbert space with a positive-definite inner product to ensure non-negative energies[1] (though this is contended[11]). Occasionally, the fourth order terms will cancel, such as in Gauss-Bonnet gravity, in which the terms involving more than two derivatives either cancel exactly, or reduce to some topological surface term. In general, however, all of these $f(R)$ theories (depending on more than a linear term in a curvature scalar or tensor) are usually beset by ghosts and unbounded Hamiltonians. It remains to be seen if there is a general solution to the problem in the context of gravity and relativity.

Cosmology/Quantum Gravity

As a final aside, I think it is worth mentioning how conformal field theory ties in. We've seen how introducing a (massless) fermion's spin connection to maintain local conformal invariance behaves exactly like the gauge connection needed to ensure local gauge invariance. As such, we can define a conformal field theory with a similar formalism to the standard quantum theory of fields with a gauge invariance prinicple. As it turns out, (super-)conformal field theory without gravity in 4 dimensions can be viewed as the boundary of a maximally symmetric space (in this case, the 5 dimensional anti-de Sitter space AdS_5). To see this, take the half-space metric for AdS_5

$$ds^2 = \frac{1}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right)$$

and perform a Weyl transformation to

$$ds^2 = dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu$$

which is Minkowski at $z = 0$. This is the famous $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondance (I've oversimplified a bit: technically we need an $\mathcal{N} = 4$ supersymmetric CFT on the boundary, and we should use $AdS_5 \times S^5$ or some other cross with a closed manifold). Hence, it is as though we can either do "regular" quantum field theory in 4 dimensions with matter fields, gauge fields, and the like, OR we may do a 5 (really 10) dimensional calculation with gravity. Which is easier usually depends on which coupling constant regime we're examining: large coupling in one space will correspond to small coupling in the other, a feature we call duality. Since anti-de Sitter (as well as de Sitter and Minkowski) space is a solution to the conformal metric, we see that a conformal structure exists on the bulk space and its boundary, indicating that conformal considerations are ubiquitous in both gravitational and quantum theories. Indeed, J. Maldacena, the physicist who first noticed the AdS/CFT correspondance, derived Einstein gravity from conformal gravity and obtained the semiclassical wavefunction of asymptotically dS or Euclidean AdS spacetimes.[5] An attempt at quantum conformal gravity has also been attempted by Mannheim with promising results.[6] The sheer universal prevalence (even if a broken symmetry) of conformal/Weyl structures in modern physics heralds its importance in future studies.

Conclusion

Demanding conformal symmetry or invariance of a metric may seem like a strict imposition, and may be problematic in having an unbounded energy spectrum, but it seems to produce results that may be consistent with our actual universe without recourse to dark matter and suggests that conformal symmetries may play an important role in dual theories and an understanding of physics beyond the standard model. We see remarkable similarities between gauge invariance and conformal invariance, and in conformal field theories on boundaries of vacuum solutions with constant curvature (AdS/CFT correspondance). Perhaps a Weyl action will prove in the end to produce spacetime solutions more consistent with large-scale gravitation than the current Einstein-Hilbert action, much as general relativity supplanted and expanded on the Newtonian view 100 years ago.

References

[1] C. M. Bender and P. D. Mannheim. No-ghost theorem for the fourth-order derivative Pais-Uhlenbeck oscillator model. *Phys.Rev.Lett.*, 100:110402, 2008, 0706.0207.

[2] D. S. Dummit and R. M. Foote. *Abstract Algebra*. Prentice Hall, Englewood Cliffs, NJ, 1991.

[3] M. Fontanini and M. Trodden. Tackling Higher Derivative Ghosts with the Euclidean Path Integral. *Phys.Rev.*, D83:103518, 2011, 1102.4357.

[4] H. M. Georgi. *Lie Algebras in Particle Physics; 2nd ed.* Frontiers in Physics. Perseus, Cambridge, 1999.

[5] J. Maldacena. Einstein Gravity from Conformal Gravity. 2011, 1105.5632.

[6] P. D. Mannheim. Conformal Gravity Challenges String Theory. 2007, 0707.2283.

[7] P. D. Mannheim. Making the Case for Conformal Gravity. 2011, 1101.2186.

[8] P. D. Mannheim and D. Kazanas. Exact Vacuum Solution to Conformal Weyl Gravity and Galactic Rotation Curves. *Astrophys. J.*, 342:635–638, 1989.

[9] P. D. Mannheim and J. G. O'Brien. Fitting galactic rotation curves with conformal gravity and a global quadratic potential. 2010, 1011.3495.

[10] K. Saifullah. Classification of spacetimes according to conformal Killing vectors. 2008, 0810.3202.

[11] A. Smilga. Comments on the dynamics of the Pais-Uhlenbeck oscillator. *SIGMA*, 5:017, 2009, 0808.0139.

[12] W. O. Straub. A Note on Weyl Conformal Gravity. 2011.

[13] R. P. Woodard. Avoiding dark energy with 1/r modifications of gravity. *Lect.Notes Phys.*, 720:403–433, 2007, astro-ph/0601672.